EXISTENCE OF DISTRIBUTION FUNCTIONS AND AVERAGE ORDER OF ARITHMETICAL FUNCTIONS(Distribution of values of arithmetic functions)

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EXISTENCE OF DISTRIBUTION FUNCTIONS AND AVERAGE ORDER OF ARITHMETICAL FUNCTIONS.

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1. Let $f(n)$ be a real sequence indexed by positive integers. f is limit-periodical B^λ, $\lambda \geq 1$, if given $\varepsilon > 0$, there exists a periodical function P_ε such that

$$\overline{\lim_{x\to+\infty}} \frac{1}{x} \sum_{n\leq x} |f(n) - P_\varepsilon(n)|^\lambda \leq \varepsilon.$$

It is known that there is a very close correspondence between B^λ and $\mathscr{L}^\lambda(\prod_p \mathbb{Z}_p, \otimes_p dm_p)$, where $\mathbb{Z}_p$ is the ring of the p-adic integers, dm_p the normalized Haar measure on $\mathbb{Z}_p$, p describing the whole set of the prime numbers (see, for instance, [1].)

2. If f belongs to B^1, f real, then, the distribution function $\sigma_f(.)$ of f exists ([2]) and moreover, if $\sigma_f(.)$ is continuous in t, then, the function $I_{t,f}(.)$ defined by

$$I_{t,f(n)} = 1 \text{ if } f(n) < t$$
$$= 0 \text{ if not}$$

is an element of B^1 ([3]).

From this, it is very easy to deduce the fact that given some real f in B^1, if there exists an $r > 1$ such that

$$\overline{\lim_{x\to+\infty}} \frac{1}{x} \sum_{n\leq x} |f(n)|^r < +\infty ,$$

then, for any λ satisfying $1 \leq \lambda < r$, we have $f \in B^{\lambda}$, (for, if σ_f is continuous in t, then $I_{t,f}(n)\, f(n)$ belongs to B^1. Now since

$$\sum_{n\leq x} |f(n) - I_{t,f}(n)f(n)|^{\lambda} = \sum_{n\leq x} |f(n)|^{\lambda}\ |1 - I_{t,f}(n)| ,$$

we get the required result by use of Hölder inequality, taking the limit for t tending to infinity.)

3. The study of elements of B^1 with values 1 or 0 only, led me very recently to the following result ([4]).

THEOREM. Let $f(n)$ be an element of B^1 with values 1 or 0 only. Suppose that $\lim_{x\to+\infty} \frac{1}{x} \sum_{n\leq x} f(n)$ is not zero. Define an arithmetical function $\delta(n)$ by :

$$\delta(n) = 0 \quad \text{if} \quad f(n) = 0$$

$$\delta(n) = \min_m \{m - n \mid m > n\} \quad \text{if} \quad f(n) = 1.$$

Then, $\delta(n) \in B^1$.

The proof is not difficult (see [4]) ; it needs in an essential way the existence of the correspondence between B^1 and $\mathcal{L}^1 (\prod_p \mathbb{Z}_p , \otimes_p dm_p)$. The other ingredients are the fact that the application $x \mapsto x+1$ from $\prod_p \mathbb{Z}_p$ to $\prod_p \mathbb{Z}_p$ is ergodic, and a (not so) well-known result of M. Kac ([5]) on Poincaré cycles. I would like to know if it is possible to get this result without the use of the correspondence between B^1 and $\mathcal{L}^1 (\prod_p \mathbb{Z}_p , \otimes_p dm_p)$. If no other method exists, then, as far as I know, this theorem would be the first of this kind.

4. Application.

I shall restrict mylself to a very special case, which is a good illustration of the usefulness of what has been explained before.

Consider the set of the square-free numbers. It was proved by Erdös ([6]) that, if $\delta(n)$ is defined by :

$\delta(n) = 0$ if n is not square-free,

$$\delta(n) = \min_{m \text{ square-free}} \{m-n \mid m > n, \ m \text{ square-free}\}$$

if n is square-free,

then : $\sum_{n \leq x} \delta(n)^{\alpha} \sim C(\alpha)x$, if $0 \leq \alpha \leq 2$, and Hooley ([7]) extended this result to the case $0 \leq \alpha \leq 3$. The existence of a distribution function for δ was proved by Mirsky ([8]).

A direct application of what has been explained before is possible, since the caracteristic function of the square-free numbers is $|\mu(n)|$, where μ is the Möbius function, which is B^1, with a not zero mean value : the theorem, in 3, gives that $\delta(n)$ is B^1 ; since δ takes only integral values, what has been stated in 2, gives that :

a. the set of the square-free numbers such that $\delta(n) = k$, $k \geq 0$ fixed, admits a density d_k and $\sum_{k \geq 0} d_k = 1$. Moreover, the caracteristic function of such a set is in B^1.

b. δ is an element of B^{λ} for any λ satisfying $1 \leq \lambda < 3$. (for $\delta(n)$ is in B^1 and $\lim_{n \to +\infty} \frac{1}{x} \sum_{n \leq x} \delta(n)^3$ exists.)

c. A necessary and sufficient condition for δ to be in B^{∞} is that

for any $\alpha \geq 0$,

$$\overline{\lim_{x \to +\infty}} \frac{1}{x} \sum_{n \leq x} \delta(n)^{\alpha} < +\infty \qquad (*)$$

(This (*) is conjectured by Erdös ([6]).)

In the same way, it is immediate that a, b, c still hold if in place of the whole set of the square-free numbers, we restrict ourself to some subset A defined by

$$A = \left\{ n \in \mathbb{N}^* \;\middle|\; \begin{array}{l} n \text{ square-free}, \\ n \equiv b \mod a \end{array}, \right\}$$

with $(a,b) = 1$, $a > b \geq 1$.

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